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A DEMONSTRATION OF MACLAURIN'S THEOREM.

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LET the following be the statement of this Theorem :

$$u = f(x) = f(0) + xf'(0) + \frac{x^2}{1.2}f''(0) + \dots + \frac{x^n}{n!}f^n(0) + \dots,$$

where $f'(x) = \frac{du}{dx}$, $f''(x) = \frac{d^2u}{dx^2}$, &c., and $f(0)$, $f'(0)$, &c. mean $f(x)$, $f'(x)$, &c., when $x = 0$.

In the demonstrations usually given of this Theorem it is necessary to know the remainder after $n + 1$ terms, to be certain that the series is convergent, that is, if

$$f(x) = f(0) + xf'(0) + \dots + \frac{x^n}{n!}f^n(0) + \frac{x^{n+1}}{n+1!}f^{n+1}(\theta x),$$

we must know that $\frac{x^{n+1}}{n+1!}f^{n+1}(\theta x)$ can be made as small as we please by sufficiently increasing n , if we would be certain that the series will approximate a correct result. This necessity makes it difficult in some cases to determine the validity of the expansion. Lagrange's and Laplace's Theorems are instances. "It must be remembered", says Mr. Todhunter, "that in quoting Maclaurin's Theorem, which serves as the foundation for those of Lagrange and Laplace, we ought strictly to have used it in the form given in Art. 95, with an expression for the remainder after $n + 1$ terms. That expression for the remainder, however, becomes so complicated in this case that we have not referred to it. The investigation of Lagrange's and Laplace's Theorems must be confessed to be imperfect, since the tests of the convergence of these series, which alone can justify our use of them as arithmetical equivalents, for the functions they profess to represent are of too difficult a character for an elementary work." (Dif. Cal., p. 115.)

The demonstration here to be given furnishes tests of the convergence of these series independent of any remainder and is therefore easily applied.

Let $f(x) = L = f(0) + L_1 = f(0) + xf'(0) + L_2 = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + L_3 = f(0) + xf'(0) + \dots + \frac{x^{n-1}}{n-1!}f^{n-1}(0) + L_n = f(0) + xf'(0) + \dots + \frac{x^{n-1}}{n-1!}f^{n-1}(0) + \frac{x^n}{n!}f^n(0) + L_{n+1}$, L_1 , L_2 , being func's of x . (1)

Prop. 1. To determine the conditions under which L_{n+1} and $f^{n+1}(0)$ have the same sign.

Let x have a certain positive value x_1 and suppose x to increase gradually in value from 0 to x_1 .

From (1), $L_n = \frac{x^n}{n!} f^n(0) + L_{n+1}$. Suppose that L_{n+1} is positive when $x = x_1$; then $L_n > \frac{x^n}{n!} f^n(0)$. But $L_n = \frac{x^n}{n!} f^n(0)$ when $x = 0$, each being equal to zero; therefore L_n has increased more with the increase of x than $\frac{x^n}{n!} f^n(0)$. It therefore either began to increase faster, began to increase at the same rate, or began to increase more slowly and afterwards increased so much faster that its increment became greater than that of $(\frac{x^n}{n!}) f^n(0)$.

Suppose that it began to increase faster. Then the first small increase of L_n is greater than that of $(\frac{x^n}{n!}) f^n(0)$. Let ΔL_n be the small increase of L_n ; $\Delta \frac{x^n}{n!} f^n(0)$, that of $\frac{x^n}{n!} f^n(0)$. Then $\Delta L_n > \Delta \frac{x^n}{n!} f^n(0)$; $\therefore$

$$\frac{\Delta L_n}{\Delta x} > \frac{\Delta (\frac{x^n}{n!}) f^n(0)}{\Delta x},$$

or, passing to the limit,

$$\frac{dL_n}{dx} > \frac{d}{dx} \frac{x^n}{n!} f^n(0); \therefore \frac{dL_n}{dx} > \frac{x^{n-1}}{(n-1)!} f^n(0).$$

But, differentiating eq. (1), $f'(x) = f'(0) + \dots + \frac{dL_n}{dx}$; $\therefore \frac{dL_n}{dx} = 0$,

when $x = 0$; $\therefore \frac{dL_n}{dx} = \frac{x^{n-1}}{(n-1)!} f^n(0)$ when $x = 0$. Therefore $\frac{dL_n}{dx}$ increases faster than $[\frac{x^{n-1}}{(n-1)!}] f^n(0)$ with the first inc. of x . Suppose we have

found that $\frac{d^r L_n}{dx^r} > \frac{x^{n-r}}{(n-r)!} f^n(0)$, r being less than n . Differentiating (1) r

times, $f^r(x) = f^r(0) + x f^{r+1}(0) + \dots + \frac{d^r L_n}{dx^r}$; $\therefore \frac{d^r L_n}{dx^r} = 0 = \frac{x^{n-r}}{(n-r)!} f^n(0)$

when $x = 0$. Therefore $\frac{d^r L_n}{dx^r}$ increases faster than $[\frac{x^{n-r}}{(n-r)!}] f^n(0)$ with the first small increase of x ; or the first small increase of $\frac{d^r L_n}{dx^r}$ is greater than that of $[\frac{x^{n-r}}{(n-r)!}] f^n(0)$. Let $\Delta(\frac{d^r L_n}{dx^r})$ be the small increase of $\frac{d^r L_n}{dx^r}$, $\Delta[\frac{x^{n-r}}{(n-r)!}] f^n(0)$ that of $[\frac{x^{n-r}}{(n-r)!}] f^n(0)$.

Then
$$\Delta \frac{d^r L_n}{dx^r} > \Delta \frac{x^{n-r}}{(n-r)!} f^n(0).$$

Dividing by Δx and passing to the limit,

$$\frac{d^{r+1} L_n}{dx^{r+1}} > \frac{d}{dx} \frac{x^{n-r}}{(n-r)!} f^n(0).$$

But we have found that

$\frac{dL_n}{dx} > \frac{x^{n-1}}{(n-1)!} f^n(0)$; $\therefore \frac{d^2 L_n}{dx^2} > \frac{x^{n-2}}{(n-2)!} f^n(0)$, $\frac{d^3 L_n}{dx^3} > \frac{x^{n-3}}{(n-3)!} f^n(0) \dots \frac{d^n L_n}{dx^n} > f^n(0)$. Differentiating eq. (1) n times, $\frac{d^n L_n}{dx^n} = f^n(x)$. Therefore $f^n(x) > f^n(0)$ on the first small increase of x . But $f^n(x) = f^n(0)$ when $x = 0$,

therefore $f^n(x)$ increases with the first small increase of x , or $f^n(x)$ begins as an increasing function of x . Therefore $f^{n+1}x$, at the beginning, when $x=0$, is pos., or $f^{n+1}(0)$ is pos. Therefore L_{n+1} and $f^{n+1}(0)$ have the same sign.

Suppose that L_n and $(x^n \div n!)f^n(0)$ begin to increase at the same rate. Then by a similar method of proof $f^n(0) = 0$, and may for all practical purposes be considered positive.

Suppose that L_n begins to increase more slowly than $(x^n \div n!)f^n(0)$. Then, by a similar method of proof, $f^n(0)$ is negative and L_n and $f^n(0)$ have different signs. But, in this case, L_n begins to grow less than $(x^n \div n!)f^n(0)$, and afterwards becomes greater. It must therefore pass through ∞ or the value of $(x^n \div n!)f^n(0)$. Therefore L_{n+1} and $f^{n+1}(0)$ have the same sign unless in the increase of x from 0 to x_1 L_n passes through ∞ or the value of $(x^n \div n!)f^n(0)$. Observe that L_n cannot pass through ∞ unless $f(x)$ does.

The same result is obtained if we suppose L_n negative.

If $f(x)$ becomes imaginary as x increases, L_{n+1} becomes imaginary also, and the preceding demonstration does not apply. In this case, L_{n+1} is neither positive nor negative.

Therefore if $f(x)$ does not become imaginary or infinite as x increases from 0 to x_1 , L_{n+1} and $f^{n+1}(0)$ have the same sign unless the value of L_n first becomes different from and then equal to that of $(x^n \div n!)f^n(0)$.

The demonstration applies and the result is equally true when $n = 0$; that is, when $L_n = L = f(x)$ and $(x^n \div n!)f^n(0) = f(0)$.

$L, L_2, \dots, L_n$, in eq. (1), may be called the remainders after 1, 2, $\dots, n$ terms. It is evident that, if each of these remainders is numerically less than the preceding one, the sum of the series in Maclaurin's Theorem continually approaches the true value of $f(x)$ as the number of terms increases. More than that, it approaches that value as long as these remainders are in a numerically decreasing order of progression, and recedes from that value whenever they are in a numerically increasing order.

Now let $f(x) = A + Bx + Cx^2 + \dots + Rx^{n-1} + Sx^n + Tx^{n+1} + \dots$ where $B = f(0)$, $C = f'(0) \div 2$, $\dots, S = f^n(0) \div n!$, &c.; and let B and L_1 , C and L_2 , S and L_n , &c, have the same sign under the conditions in Prop. 1.

1°. Let all the terms after Rx^{n-1} have the same sign. By equation (1), $L_{n-1} = Rx^{n-1} + L_n$, $L_n = Sx^n + L_{n+1}$, &c. Therefore L_n is numerically less than L_{n-1} , L_{n+1} less than L_n , &c.; and the sum of the series continually approaches the value of $f(x)$. This general result is equally true if any term, as Rx^{n-1} , vanishes.

2°. Let the terms after Rx^{n-1} have alternate signs. $L_{n-1} = Rx^{n-1} + Sx^n + Tx^{n+1}$. L_{n-1} , Rx^{n-1} and Tx^{n+1} have the same sign, Sx^n a diff. sign.

Therefore if Rx^{n-1} is numerically greater than Sx^n , L_{n+1} is numerically less than L_{n-1} , and *vice versa*. So with the succeeding remainders. Therefore the sum of a decreasing series of this kind, or of a series of this kind while it is decreasing, approaches the value of $f(x)$; that of an increasing series, or a series while it is increasing, recedes from that value.

This result is equally true if any number of terms vanish between each positive and negative sign.

In this case we can determine the nearness of the approximation at any period. For $L_n = Sx^n - L_{n+1}$. Therefore Sx^n is greater than either L_n or L_{n+1} ; that is the remainder after n terms is less than Sx^n . If the limit of Sx^n when n is indefinitely large, is zero, the limit of the remainder is also zero, and $f(x)$ is equal to the sum of the series

3°. Let a positive signs be followed by b negative signs, these by c positive signs, these by d negative signs, &c. Let

$f(x) = A + \dots + Rx^m + \dots - Sx^n - \dots + Tx^r + \dots - Ux^s - \dots$, the number of terms after Rx^m being $a-1$, after Sx^n , $b-1$, after Tx^r , $c-1$, &c. $L_n = Rx^m + \dots - Sx^n - \dots + L_r$. Therefore, if $Rx^m + \dots$ is numerically greater than $Sx^n + \dots$, L_r is numerically less than L_m and *vice versa*. So if $Sx^n + \dots$ is numerically greater than $Tx^r + \dots$, L_s is numerically less than L_n , and *vice versa*. Therefore, if the sum of the succeeding negative or positive terms is numerically less than that of the preceding positive or negative terms, the sum of the series approaches the value of $f(x)$, and *vice versa*.

This result is equally true if some terms vanish. In this case, as in the preceding, we can determine the nearness of the approximation at the point where the signs change, and, if the limit of Sx^n is zero $f(x)$ is equal to the sum of the series indefinitely extended.

When the signs of the terms are the same, we have not proved that $f(x)$ is equal to the sum of the series, for, although $L, L_2 \dots L_n$, &c., continually decrease, their limit may not be zero. It is evident, however, that $f(x)$ is either equal to, or numerically greater than, the sum of the series. Their equality will be proved hereafter.

If L_n does not become different from Sx^n as x increases, $f(x) = A + Bx + \dots + Sx^n$. Moreover, as

$$L_n = Sx^n = \frac{x^n}{n!} f^n(0), \quad \frac{d^n L_n}{dx^n} = f^n(0);$$

therefore $f^n(x) = f^n(0)$, a constant, and the succeeding terms vanish.

Suppose now that $f(x)$ becomes neither imaginary nor infinite as x increases from 0 to x_1 ; then $f(x)$ is continuous and L_n is also continuous; that is, for an indefinite increase of x there is a corresponding indefinite

change of $f(x)$ or L_n . Let x_2 be some value of x such that $L_n, L_{n+1}, \&c.$ have become different in value from $Sx^n, Tx^{n+1}, \&c.$, as x increases from 0 to x_2 but not equal to them again, and such that $A + Bx_2 + \&c.$ is a decreasing series and therefore falls under one of the conditions in 1°, 2°, 3°.

It is evident that there is such a value of x unless L_n or some succeeding remainder is always equal to Sx^n or some succeeding term, in which case, as we have seen, the series terminates, and $f(x)$ under any circumstance is equal to the sum of the series. Then $f(x_2)$ is equal to, or numerically greater than $A + Bx_2 + \&c.$; and L_n is equal to, or numerically greater than $Sx_2^n + \&c.$, continued indefinitely. Or, it need not be continued indefinitely; for $f(x)$ is numerically greater than $A + Bx_2 + \&c.$, continued to a term with the same sign that $f(x)$ has, and L_n is greater than $Sx_2^n + \&c.$, continued to a term with the same sign that L_n has. Now suppose that x gradually increases from x_2 , under one of the conditions established in 1°, 2°, 3°. L_n will continue equal to, or numerically greater than $Sx^n + \&c.$, continued as before till L_n or some succeeding remainder becomes equal to Sx^n or some succeeding term. But L_n gradually changes with the gradual increase of x , and therefore cannot become equal to Sx^n till it first becomes indefinitely near to that value; and so with any succeeding remainder. But, in this case, the equal or smaller quantity $Sx^n + \&c.$, continued as before, becomes, or has become indefinitely near to Sx^n , or the sum of the terms succeeding Sx^n becomes, or has become, indefinitely near zero. Under the conditions in 1°, 2°, 3°, therefore, L_n cannot pass through the value of Sx^n till the sum of the succeeding terms, continued indefinitely or to a term with the sign of L_n , has become indefinitely near to zero.

Now suppose that all the terms after Sx^n have the same sign. Then $Sx^n + \&c.$ continually increase numerically with the increase of x . It never becomes indefinitely near to Sx^n ; therefore L_n never passes through the value of Sx^n . So with any succeeding remainder.

Suppose that the signs alternate, and the series decreases after Sx^n for the value of x_1 . L_n cannot pass through Sx^n till $Tx^{n+1} - Ux^{n+2} + Vx^{n+3} + Wx^{n+4} \&c.$, or $x^{n+1}(T - Ux) + x^{n+3}(V - Wx) + \&c.$, becomes indefinitely near to zero. $T - Ux, V - Wx, \&c.$ have the same sign and, after the value x_1 are not indefinitely near to zero. Moreover, for any value of x less than x_1 $T - Ux, V - Wx, \&c.$, are numerically greater than $T - Ux_1, V - Wx_1, \&c.$ On the gradual increase of x from 0 to x_1 , then, neither $x, T - Ux, V - Wx, \&c.$, has become indefinitely near to zero. Therefore L_n has not passed through the value of Sx^n . So with any succeeding remainder.

In the same way we can prove that L_n has not passed through the value of Sx^n in case 3°.

This result, it is evident, is equally true if some terms vanish.

In the first part of this demonstration we had the condition that L_n must not pass through the value of Sx^n . But as we have since proved that this cannot be the case under the conditions in 1^o , 2^o , 3^o , we may erase this condition and we have the following

General Result. In the expansion of $f(x)$ by Maclaurin's Theorem, unless $f(x)$ becomes imaginary or infinite as x increases from 0 to x_1 the sum of the series approaches the true value of $f(x)$ —

1. If the signs of the terms are the same;
2. If the signs alternate, and the series decreases;
3. If m terms of one sign follow n terms of the other, and the sum of the former is numerically less than that of the latter.
4. If the series terminates.

[To be continued.]

SOME EXAMPLES OF A NEW METHOD OF SOLVING PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

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EXAMPLE 1. It is required to formulate a method for integrating partial differential equations of the second order, with variable coefficients, that shall facilitate their integration from the terms of the first order. It is also required to apply the method to the integration of the equation

$$\frac{d^2u}{dt^2} - \frac{dx^2}{x^2} = \frac{du}{dt} - \frac{dx}{x^3} + P.$$

Solution.—First, as to the required method. Let u be any function of t and x . Then since

$$\frac{du}{dt} = \left(\frac{du}{dt}\right) + \frac{du}{dx} \cdot \frac{dx}{dt},$$

let it be put in the symbolic form

$$D_t u = d_t u + D_x \cdot d_x u. \tag{\alpha}$$

Let equation (α) be differentiated with the symbol $\mathcal{A}$, so that

$$\mathcal{A}_t D_t u = \mathcal{A}_t(d_t u) + \mathcal{A}_t(D_x \cdot d_x u) + D_x \cdot \mathcal{A}_t(d_x u). \tag{\beta}$$

Now if in equation (α) we place $d_t u$ and $d_x u$ successively for u , we shall find